On the Horadam hybrid quaternions

Ali Dağdeviren[1] and Ferhat Kürüz[2]

[1]Khoja Akhmet Yassawi International Kazak-Turkish University, Faculty of Engineering, Department of Computer Engineering, Turkistan, Kazakhstan
[2]Istanbul Gelisim University, Faculty of Engineering and Architecture, Department of Computer Engineering, Istanbul, Turkey

E-mail: ali.dagdeviren@ayu.edu.kz[1], m.a.dagdeviren@gmail.com[1], fkuruz@gelisim.edu.tr[2]

Abstract

In this study, we firstly define the Horadam hybrid quaternions and present some of their properties. Then, we define Fibonacci and Lucas hybrid quaternions, and also we study the relationship between the Fibonacci and the Lucas hybrid quaternions which connect the Fibonacci quaternions and Lucas quaternions. Furthermore, we also give some identities such as the Binet formulas and Cassini identities for Fibonacci and Lucas hybrid quaternions.

2010 Mathematics Subject Classification. **11B39**. 11R52.
Keywords. Horadam numbers, hybrid numbers, fibonacci numbers, hybrid quaternions.

1 Introduction

Quaternions were introduced by Sir William Rowan Hamilton in 1866 as an extension of the complex numbers [1]. Quaternions are an important number system used in different areas such as computer science, quantum physics, and analysis [2, 3, 4]. This type of quaternions also called real quaternions. A real quaternion is defined as

$$Q = z_0 + z_1 i + z_2 j + z_3 k$$

where z_0, z_1, z_2, z_3 are real numbers. Also, i, j, and k are the units of real quaternions which satisfy the following equalities

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1.1}$$

Note that the set of real quaternions form non-commutative but associative algebra. The conjugate of the quaternion Q is defined by $\overline{Q} = z_0 - z_1 i - z_2 j - z_3 k$. Moreover, the norm of any quaternion Q is denoted by $\|Q\|$ and defined by $\|Q\| = \sqrt{Q\overline{Q}} = \sqrt{z_0^2 + z_1^2 + z_2^2 + z_3^2} \in \mathbb{R}$. For further information about real quaternions see [5].

Hybrid numbers have been defined by Özdemir [6]. In this work, a number system of such numbers consisting of all three number systems(complex, dual, and hyperbolic) has been given. The set of hybrid numbers, $\mathbb{K}$, is defined as following:

$$\mathbb{K} = \left\{ z = a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h} : \ a, b, c, d \in \mathbb{R}, \ \begin{array}{c} \mathbf{i}^2 = -1, \boldsymbol{\varepsilon}^2 = 0, \mathbf{h}^2 = 1 \\ \mathbf{ih} = -\mathbf{hi} = \boldsymbol{\varepsilon} + \mathbf{i} \end{array} \right\}. \tag{1.2}$$

Multiplication rules of $\mathbf{i}, \boldsymbol{\varepsilon}$, and $\mathbf{h}$ can be given as following table:

Advanced Studies: Euro-Tbilisi Mathematical Journal Special Issue (10 - 2022), pp. 293–306.

Tbilisi Centre for Mathematical Sciences.

Received by the editors: 26 May 2021.
Accepted for publication: 28 October 2021.

.	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
$\mathbf{i}$	-1	$1-\mathbf{h}$	$\boldsymbol{\varepsilon}+\mathbf{i}$
$\boldsymbol{\varepsilon}$	$1+\mathbf{h}$	0	$-\boldsymbol{\varepsilon}$
$\mathbf{h}$	$-\boldsymbol{\varepsilon}-\mathbf{i}$	$\boldsymbol{\varepsilon}$	1

Table 1. Multiplication table of hybrid units

The conjugate of the hybrid number is defined by $z^c = a - b\mathbf{i} - c\boldsymbol{\varepsilon} - d\mathbf{h}$. Furthermore, the norm of any hybrid number z is

$$\|z\| = \sqrt{zz^c} = \sqrt{a^2 + (b-c)^2 - c^2 - d^2}.$$

For further information about hybrid number system, see [6].

The hybrid quaternions have recently been defined as a new quaternion system by Dağdeviren in [7]. This system has a strong algebraic structure and it is a generalization of complex, dual, and hyperbolic quaternions. That is, complex quaternions, dual quaternions and hyperbolic quaternions can be obtained from hybrid quaternions in special cases. Moreover, hybrid quaternions are also generalised the features of the other three quaternion systems such as inner product, vector product, and norm. The set of hybrid quaternions denoted by $\mathbf{H}_{\mathbb{K}}$ and defined as

$$\mathbf{H}_{\mathbb{K}} = \{Q = z_0 + z_1 i + z_2 j + z_3 k \mid z_0, z_1, z_2, z_3 \in \mathbb{K}\} \tag{1.3}$$

where quaternionic units i, j, k satisfies the equation (1.1). The quaternionic units i, j, k commutes with the hybrid units $\mathbf{i}, \boldsymbol{\varepsilon}, \mathbf{h}$. Thus, any hybrid quaternion can be written as

$$Q = q_0 + q_1\mathbf{i} + q_2\boldsymbol{\varepsilon} + q_3\mathbf{h}$$

where q_0, q_1, q_2, q_3 are quaternions and $\mathbf{i}, \boldsymbol{\varepsilon}, \mathbf{h}$ are hybrid units obeying the multiplication rules in the Table 1, [7].

Two hybrid quaternions are equal if all their components are equal, one by one. Let $Q = q_0 + q_1\mathbf{i} + q_2\boldsymbol{\varepsilon} + q_3\mathbf{h}$ and $P = p_0 + p_1\mathbf{i} + p_2\boldsymbol{\varepsilon} + p_3\mathbf{h}$ and be any two hybrid quaternions. Addition and subtraction of these two hybrid quaternions are defined as

$$Q \mp P = (q_0 \mp p_0) + (q_1 \mp p_1)\mathbf{i} + (q_2 \mp p_2)\boldsymbol{\varepsilon} + (q_3 \mp p_3)\mathbf{h}.$$

Multiplication of the hybrid quaternions is defined as

$$\begin{aligned} QP = {} & (q_0p_0 - q_1p_1 + q_3p_3 + q_1p_2 + q_2p_1) \\ & + (q_0p_1 + q_1p_0 + q_1p_3 - q_3p_1)\ \mathbf{i} \\ & + (q_0p_2 + q_2p_0 + q_1p_3 - q_3p_1 + q_3p_2 - q_2p_3)\ \boldsymbol{\varepsilon} \\ & + (q_0p_3 + q_3p_0 + q_2p_1 - q_1p_2)\ \mathbf{h}\ . \end{aligned}$$

The multiplication of any two hybrid numbers with the help of hybrid units is expressed as above. Furthermore, the hybrid quaternions Q and P above can be written as $Q = z_0 + z_1 i + z_2 j + z_3 k$ and $P = t_0 + t_1 i + t_2 j + t_3 k$ in terms of quaternion units. Multiplication of these two hybrid quaternions can be given as

$$\begin{aligned} QP = {} & (z_0t_0 - z_1t_1 - z_2t_2 - z_3t_3) \\ & + (z_1t_0 + z_0t_1 - z_3t_2 + z_2t_3)\ i \\ & + (z_2t_0 + z_3t_1 + z_0t_2 - z_1t_3)\ j \\ & + (z_3t_0 - z_2t_1 + z_1t_2 + z_0t_3)\ k\ . \end{aligned}$$

Further information about hybrid quaternions can be found in [7].

Generalized Fibonacci numbers were defined by Horadam [8, 9]. Moreover Horadam gave the formula for negative index terms of Horadam numbers [9]. Horadam numbers are defined as follow:

$$w_n = pw_{n-1} - qw_{n-2}\ ,\ n \geq 2 \tag{1.4}$$

where p, q, n are integers and w_0, w_1 are initial conditions. The Binet's formula of the Horadam numbers is

$$w_n = A\alpha^n + B\beta^n \tag{1.5}$$

where α and β are the roots of the equation $x^2 - px + q = 0$. Also A and B are

$$A = \frac{w_1 - w_0\beta}{\alpha - \beta}\ ,\ B = \frac{w_0\alpha - w_1}{\alpha - \beta}\quad . \tag{1.6}$$

In addition, Horadam numbers can be represent as $w_n(w_0, w_1; p, q)$. For special w_0, w_1, p and q the equation (1.4) defines the well known numbers named as the Fibonacci type numbers. These types of numbers can be listed as follows:

i) $w_n(0,1;p,q) = \mathcal{U}_n-$ Generalized Fibonacci numbers,

ii) $w_n(2,1;p,q) = \mathcal{V}_n-$ Generalized Lucas numbers,

iii) $w_n(0,1;1,-1) = \mathcal{F}_n-$ Fibonacci numbers,

iv) $w_n(2,1;1,-1) = \mathcal{L}_n-$ Lucas numbers,

v) $w_n(0,1;2,-1) = \mathcal{P}_n-$ Pell numbers,

vi) $w_n(2,2;2,-1) = \mathcal{PL}_n-$ Pell-Lucas numbers,

vii) $w_n(0,1;1,-2) = \mathcal{J}_n -$ Jacobsthal numbers,

viii) $w_n(2,1;1,-2) = \mathcal{JL}_n-$ Jacobsthal-Lucas numbers,

ix) $w_n(0,1;3,\ 2) = \mathcal{M}_n-$ Mersenne Numbers,

x) $w_n(1,3;3,-2) = \mathcal{FE}_n-$ Fermat numbers.

These numbers have been studied from different perspectives [10, 11, 12, 13, 14, 9, 15, 16]. Horadam hybrid numbers, a special type of hybrid numbers, were introduced and studied by Szynal-Liana [11]. In this study, the author gave the Binet formulas, generating functions and characters for Horadam hybrid numbers. nth Horadam hybrid number is defined as

$$\mathcal{HH}_n = w_n + \mathbf{i}\ w_{n+1} + \varepsilon\ w_{n+2} + \mathbf{h}\ w_{n+3} \tag{1.7}$$

where w_n is nth Horadam number. The Binet formula of the Horadam hybrid numbers is

$$w_n = A\alpha^*\alpha^n + B\beta^*\beta^n \tag{1.8}$$

where A, B are defined by (1.6) and $\alpha^* = 1 + \mathbf{i}\alpha + \boldsymbol{\varepsilon}\alpha^2 + \mathbf{h}\alpha^3$, $\beta^* = 1 + \mathbf{i}\beta + \boldsymbol{\varepsilon}\beta^2 + \mathbf{h}\beta^3$. Moreover, in the same article, the author defined Fibonacci and Lucas hybrid numbers as follows:

$$\mathcal{FH}_n = \mathcal{F}_n + \mathbf{i}\mathcal{F}_{n+1} + \boldsymbol{\varepsilon}\mathcal{F}_{n+2} + \mathbf{h}\mathcal{F}_{n+3},$$
$$\mathcal{LH}_n = \mathcal{L}_n + \mathbf{i}\mathcal{L}_{n+1} + \boldsymbol{\varepsilon}\mathcal{L}_{n+2} + \mathbf{h}\mathcal{L}_{n+3}.$$

The other studies about Horadam hybrid numbers can be found in [17, 18, 19, 20, 21, 22].

In the literature, there are also studies on Fibonacci quaternions. In [8], nth Fibonacci and Lucas quaternions were defined by Horadam as follows:

$$\mathcal{FQ}_n = \mathcal{F}_n + i\mathcal{F}_{n+1} + j\mathcal{F}_{n+2} + k\mathcal{F}_{n+3},$$
$$\mathcal{LQ}_n = \mathcal{L}_n + i\mathcal{L}_{n+1} + j\mathcal{L}_{n+2} + k\mathcal{L}_{n+3}$$

where i, j and k are quaternion units which satisfy equations (1.1). Moreover, $\mathcal{F}_n$ and $\mathcal{L}_n$ are the nth Fibonacci and Lucas numbers, respectively. In [14], Halc have studied on Fibonacci quaternions and present the generating functions and Binet formulas for Fibonacci and Lucas quaternions. For the other studies about Fibonacci and Lucas quaternions see [23, 14].

In this study, using the hybrid quaternions firstly we will introduce the Horadam Hybrid quaternions to the literature by defining a more general structure of Horadam hybrid numbers and Horadam quaternions. Then, we will give Binet formulas for these numbers. Furthermore, we will examine the Fibonacci and Lucas hybrid quaternions in detail and consequently, we will give some properties and identities of these numbers.

In what follows, to avoid confusion we use notation as properly as we can. We will give the following table to make the symbols used in this study easier to understand.

Notations	**Numbers**
$\mathcal{FHQ}_n$	Fibonacci hybrid quaternions
$\mathcal{LHQ}_n$	Lucas hybrid quaternions
$\mathcal{FQ}_n$	Fibonacci quaternions
$\mathcal{LQ}_n$	Lucas quaternions
$\mathcal{FH}_n$	Fibonacci hybrid numbers
$\mathcal{LH}_n$	Lucas hybrid numbers
$\mathcal{F}_n$	Fibonacci numbers
$\mathcal{L}_n$	Lucas numbers

Table 2. Notation table of Numbers

2 Horadam hybrid quaternions

In this section, we define Horadam hybrid quaternions by using Horadam numbers. Therefore, we define the Horadam hybrid quaternion $\mathcal{HHQ}_n$ as

$$\mathcal{HHQ}_n = \mathcal{HH}_n + i\mathcal{HH}_{n+1} + j\mathcal{HH}_{n+2} + k\mathcal{HH}_{n+3} \tag{2.1}$$

where i, j, k are quaternion units which satisfy equations (1.1) and $\mathcal{HH}_n$ is the nth Horadam hybrid number as in (1.7). Moreover, every Horadam hybrid quaternion $\mathcal{HQ}_n$ can be written as

$$\begin{aligned}\mathcal{HHQ}_n &= (w_n + \mathbf{i}w_{n+1} + \boldsymbol{\varepsilon}w_{n+2} + \mathbf{h}w_{n+3}) + (w_{n+1} + \mathbf{i}w_{n+2} + \boldsymbol{\varepsilon}w_{n+3} + \mathbf{h}w_{n+4})\ i \\ &\quad + (w_{n+2} + \mathbf{i}w_{n+3} + \boldsymbol{\varepsilon}w_{n+4} + \mathbf{h}w_{n+5})\ j + (w_{n+3} + \mathbf{i}w_{n+4} + \boldsymbol{\varepsilon}w_{n+5} + \mathbf{h}w_{n+6})\ k \\ &= \mathcal{HQ}_n + \mathbf{i}\mathcal{HQ}_{n+1} + \boldsymbol{\varepsilon}\mathcal{HQ}_{n+2} + \mathbf{h}\mathcal{HQ}_{n+3}\end{aligned}$$

where $\mathbf{i}, \boldsymbol{\varepsilon}, \mathbf{h}$ are hybrid units and $\mathcal{HQ}_n = w_n + iw_{n+1} + jw_{n+2} + kw_{n+3}$ is the nth Horadam quaternion. The Fibonacci, Lucas, Pell, Jacobsthal, Pell-Lucas, and Jacobsthal-Lucas hybrid quaternions can be defined both by using Fibonacci($\mathcal{FH}_n$), Lucas($\mathcal{LH}_n$), Pell($\mathcal{PH}_n$), Jacobsthal($\mathcal{JH}_n$), Pell-Lucas($\mathcal{PLH}_n$), and Jacobsthal-Lucas($\mathcal{JLH}_n$) hybrid number coefficients and by using Fibonacci ($\mathcal{FQ}_n$), Lucas($\mathcal{LQ}_n$), Pell($\mathcal{PQ}_n$), Jacobsthal($\mathcal{JQ}_n$), Pell-Lucas($\mathcal{PLQ}_n$), and Jacobsthal-Lucas($\mathcal{JLQ}_n$) quaternion coefficients respectively. We will define these hybrid quaternions as follows:

i) nth Fibonacci hybrid quaternion $\mathcal{FHQ}_n$ is

$$\begin{aligned}\mathcal{FHQ}_n &= \mathcal{FH}_n + i\mathcal{FH}_{n+1} + j\mathcal{FH}_{n+2} + k\mathcal{FH}_{n+3} \\ &= \mathcal{FQ}_n + \mathbf{i}\mathcal{FQ}_{n+1} + \boldsymbol{\varepsilon}\mathcal{FQ}_{n+2} + \mathbf{h}\mathcal{FQ}_{n+3},\end{aligned}$$

ii) nth Lucas hybrid quaternion $\mathcal{LHQ}_n$ is

$$\begin{aligned}\mathcal{LHQ}_n &= \mathcal{LH}_n + i\mathcal{LH}_{n+1} + j\mathcal{LH}_{n+2} + k\mathcal{LH}_{n+3} \\ &= \mathcal{LQ}_n + \mathbf{i}\mathcal{LQ}_{n+1} + \boldsymbol{\varepsilon}\mathcal{LQ}_{n+2} + \mathbf{h}\mathcal{LQ}_{n+3},\end{aligned}$$

iii) nth Pell hybrid quaternion $\mathcal{PHQ}_n$ is

$$\begin{aligned}\mathcal{PHQ}_n &= \mathcal{PH}_n + i\mathcal{PH}_{n+1} + j\mathcal{PH}_{n+2} + k\mathcal{PH}_{n+3} \\ &= \mathcal{PQ}_n + \mathbf{i}\mathcal{PQ}_{n+1} + \boldsymbol{\varepsilon}\mathcal{PQ}_{n+2} + \mathbf{h}\mathcal{PQ}_{n+3},\end{aligned}$$

iv) nth Jacobsthal hybrid quaternion $\mathcal{JHQ}_n$ is

$$\begin{aligned}\mathcal{JHQ}_n &= \mathcal{JH}_n + i\mathcal{JH}_{n+1} + j\mathcal{JH}_{n+2} + k\mathcal{JH}_{n+3} \\ &= \mathcal{JQ}_n + \mathbf{i}\mathcal{JQ}_{n+1} + \boldsymbol{\varepsilon}\mathcal{JQ}_{n+2} + \mathbf{h}\mathcal{JQ}_{n+3},\end{aligned}$$

v) nth Pell-Lucas hybrid quaternion $\mathcal{PLHQ}_n$ is

$$\begin{aligned}\mathcal{PLHQ}_n &= \mathcal{PLH}_n + i\mathcal{PLH}_{n+1} + j\mathcal{PLH}_{n+2} + k\mathcal{PLH}_{n+3} \\ &= \mathcal{PLQ}_n + \mathbf{i}\mathcal{PLQ}_{n+1} + \boldsymbol{\varepsilon}\mathcal{PLQ}_{n+2} + \mathbf{h}\mathcal{PLQ}_{n+3},\end{aligned}$$

vi) nth Jacobsthal-Lucas hybrid quaternion $\mathcal{JLHQ}_n$ is

$$\begin{aligned}\mathcal{JLHQ}_n &= \mathcal{JLH}_n + i\mathcal{JLH}_{n+1} + j\mathcal{JLH}_{n+2} + k\mathcal{JLH}_{n+3} \\ &= \mathcal{JLQ}_n + \mathbf{i}\mathcal{JLQ}_{n+1} + \boldsymbol{\varepsilon}\mathcal{JLQ}_{n+2} + \mathbf{h}\mathcal{JLQ}_{n+3}.\end{aligned}$$

Theorem 2.1. Let $n \in \mathbb{N}$, then the Binet formula for the Horadam hybrid quaternions is

$$\mathcal{HHQ}_n = A\alpha^* \underline{\alpha}\alpha^n + B\beta^* \underline{\beta}\beta^n$$

where A, B are defined by (1.6) and

$$\begin{aligned} \alpha^* &= 1 + \mathbf{i}\alpha + \boldsymbol{\varepsilon}\alpha^2 + h\alpha^3 \quad , \quad \beta^* = 1 + \mathbf{i}\beta + \boldsymbol{\varepsilon}\beta^2 + h\beta^3, \\ \underline{\alpha} &= 1 + i\alpha + j\alpha^2 + k\alpha^3 \quad , \quad \underline{\beta} = 1 + i\beta + j\beta^2 + k\beta^3. \end{aligned}$$

Proof. By using the definition of Horadam hybrid quaternions (2.1) and the Binet formula for the Horadam hybrid numbers (1.8), we obtain

$$\begin{aligned} \mathcal{HHQ}_n &= (A\alpha^n\alpha^* + B\beta^n\beta^*) + i(A\alpha^{n+1}\alpha^* + B\beta^{n+1}\beta^*) \\ &\quad + j(A\alpha^{n+2}\alpha^* + B\beta^{n+2}\beta^*) + k(A\alpha^{n+3}\alpha^* + B\beta^{n+3}\beta^*) \\ &= (A\alpha^n\alpha^*)(1 + i\alpha + j\alpha^2 + k\alpha^3) + (B\beta^n\beta^*)(1 + i\beta + j\beta^2 + k\beta^3) \\ &= A\alpha^*\underline{\alpha}\alpha^n + B\beta^*\underline{\beta}\beta^n. \end{aligned}$$

Q.E.D.

3 Fibonacci and Lucas hybrid quaternions

In this section, we examine the Fibonacci and Lucas Hybrid quaternions in detail and give some properties.

Definition 3.1. We denote the set of Fibonacci hybrid quaternios by FHQ and define as follows:

$$FHQ = \{\mathcal{FHQ}_n = \mathcal{FH}_n + i\mathcal{FH}_{n+1} + j\mathcal{FH}_{n+2} + k\mathcal{FH}_{n+3} \mid \mathcal{FH}_n, n\text{th Fibonacci hybrid number}\}$$

where i, j, k are quaternionic units. Moreover, here nth, $(n+1)th$, $(n+2)th$ and $(n+3)th$ Fibonacci hybrid numbers are

$$\mathcal{FH}_n = \mathcal{F}_n + \mathbf{i}\mathcal{F}_{n+1} + \boldsymbol{\varepsilon}\mathcal{F}_{n+2} + \mathbf{h}\mathcal{F}_{n+3}, \tag{3.1}$$
$$\mathcal{FH}_{n+1} = \mathcal{F}_{n+1} + \mathbf{i}\mathcal{F}_{n+2} + \boldsymbol{\varepsilon}\mathcal{F}_{n+3} + \mathbf{h}\mathcal{F}_{n+4}, \tag{3.2}$$
$$\mathcal{FH}_{n+2} = \mathcal{F}_{n+2} + \mathbf{i}\mathcal{F}_{n+3} + \boldsymbol{\varepsilon}\mathcal{F}_{n+4} + \mathbf{h}\mathcal{F}_{n+5}, \tag{3.3}$$
$$\mathcal{FH}_{n+3} = \mathcal{F}_{n+3} + \mathbf{i}\mathcal{F}_{n+4} + \boldsymbol{\varepsilon}\mathcal{F}_{n+5} + \mathbf{h}\mathcal{F}_{n+6} \tag{3.4}$$

where $\mathbf{i}, \boldsymbol{\varepsilon}$, and $\mathbf{h}$ are the hybrid units. We will restate $\mathcal{FHQ}_n$ by using the equations (3.1), (3.2), (3.3) and (3.4) as below.

$$\begin{aligned} \mathcal{FHQ}_n &= (\mathcal{F}_n + \mathbf{i}\mathcal{F}_{n+1} + \boldsymbol{\varepsilon}\mathcal{F}_{n+2} + \mathbf{h}\mathcal{F}_{n+3}) + i(\mathcal{F}_{n+1} + \mathbf{i}\mathcal{F}_{n+2} + \boldsymbol{\varepsilon}\mathcal{F}_{n+3} + \mathbf{h}\mathcal{F}_{n+4}) \\ &\quad + j(\mathcal{F}_{n+2} + \mathbf{i}\mathcal{F}_{n+3} + \boldsymbol{\varepsilon}\mathcal{F}_{n+4} + \mathbf{h}\mathcal{F}_{n+5}) + k(\mathcal{F}_{n+3} + \mathbf{i}\mathcal{F}_{n+4} + \boldsymbol{\varepsilon}\mathcal{F}_{n+5} + \mathbf{h}\mathcal{F}_{n+6}) \\ \mathcal{FHQ}_n &= \mathcal{FQ}_n + \mathbf{i}\mathcal{FQ}_{n+1} + \boldsymbol{\varepsilon}\mathcal{FQ}_{n+2} + \mathbf{h}\mathcal{FQ}_{n+3} \end{aligned}$$

where $\mathcal{FQ}_n = \mathcal{F}_n + i\mathcal{F}_{n+1} + j\mathcal{F}_{n+2} + k\mathcal{F}_{n+3}$ is a Fibonacci quaternion. Therefore, the set of Fibonacci hybrid quaternios FHQ can be redefined as

$$FHQ = \left\{\mathcal{FHQ}_n = \mathcal{FQ}_n + \mathbf{i}\mathcal{FQ}_{n+1} + \boldsymbol{\varepsilon}\mathcal{FQ}_{n+2} + \mathbf{h}\mathcal{FQ}_{n+3} \mid \begin{array}{c}\mathcal{FQ}_n, n\text{th Fibonacci} \\ \text{quaternion}\end{array}\right\}. \tag{3.5}$$

Remark 3.2. Every Fibonacci hybrid quaternion

$$\mathcal{FHQ}_n = \mathcal{FH}_n + i\mathcal{FH}_{n+1} + j\mathcal{FH}_{n+2} + k\mathcal{FH}_{n+3}$$

can be written as

$$\mathcal{FHQ}_n = \mathcal{FQ}_n + \mathbf{i}\mathcal{FQ}_{n+1} + \varepsilon\mathcal{FQ}_{n+2} + \mathbf{h}\mathcal{FQ}_{n+3}.$$

Definition 3.3. We denote the set of Lucas hybrid quaternios by LHQ and define as

$$LHQ = \left\{ \mathcal{LHQ}_n = \mathcal{LH}_n + i\mathcal{LH}_{n+1} + j\mathcal{LH}_{n+2} + k\mathcal{LH}_{n+3} \mid \begin{array}{c} \mathcal{LH}_n, n\text{th Lucas} \\ \text{hybrid number} \end{array} \right\} \tag{3.6}$$

where i, j, k are quaternionic units. Moreover, here $\mathcal{LH}_n$ is nth Lucas hybrid number

$$\mathcal{LH}_n = \mathcal{L}_n + \mathbf{i}\mathcal{L}_{n+1} + \varepsilon\mathcal{L}_{n+2} + \mathbf{h}\mathcal{L}_{n+3}$$

As with Fibonacci hybrid quaternions above, Lucas hybrid quaternions can be redefined in terms of Lucas quaternions by

$$LHQ = \left\{ \mathcal{LHQ}_n = \mathcal{LQ}_n + \mathbf{i}\mathcal{LQ}_{n+1} + \varepsilon\mathcal{LQ}_{n+2} + \mathbf{h}\mathcal{LQ}_{n+3} \middle| \begin{array}{c} \mathcal{LQ}_n, n\text{th Lucas} \\ \text{quaternion} \end{array} \right\} \tag{3.7}$$

where $\mathbf{i}, \varepsilon, \mathbf{h}$ are hybrid units. Moreover, here $\mathcal{LQ}_n$ is nth Lucas quaternion

$$\mathcal{LQ}_n = \mathcal{L}_n + i\mathcal{L}_{n+1} + j\mathcal{L}_{n+2} + k\mathcal{L}_{n+3}.$$

Remark 3.4. Every Lucas hybrid quaternion

$$\mathcal{LHQ}_n = \mathcal{LH}_n + i\mathcal{LH}_{n+1} + j\mathcal{LH}_{n+2} + k\mathcal{LH}_{n+3}$$

can be written as

$$\mathcal{LHQ}_n = \mathcal{LQ}_n + \mathbf{i}\mathcal{LQ}_{n+1} + \varepsilon\mathcal{LQ}_{n+2} + \mathbf{h}\mathcal{LQ}_{n+3}.$$

Definition 3.5. Let $\mathcal{FHQ}_n$ and $\mathcal{FHQ}_m$ be nth and mth terms of the Fibonacci hybrid quaternion sequences such that

$$\begin{aligned} \mathcal{FHQ}_n &= \mathcal{FH}_n + i\mathcal{FH}_{n+1} + j\mathcal{FH}_{n+2} + k\mathcal{FH}_{n+3} \\ &= \mathcal{FQ}_n + \mathbf{i}\mathcal{FQ}_{n+1} + \varepsilon\mathcal{FQ}_{n+2} + \mathbf{h}\mathcal{FQ}_{n+3} \end{aligned}$$

and

$$\begin{aligned} \mathcal{FHQ}_m &= \mathcal{FH}_m + i\mathcal{FH}_{m+1} + j\mathcal{FH}_{m+2} + k\mathcal{FH}_{m+3} \\ &= \mathcal{FQ}_m + \mathbf{i}\mathcal{FQ}_{m+1} + \varepsilon\mathcal{FQ}_{m+2} + \mathbf{h}\mathcal{FQ}_{m+3}. \end{aligned}$$

Then, the addition and subtraction of the Fibonacci hybrid quaternions are defined by

$$\begin{aligned} \mathcal{FHQ}_n \mp \mathcal{FHQ}_m &= (\mathcal{FH}_n + i\mathcal{FH}_{n+1} + j\mathcal{FH}_{n+2} + k\mathcal{FH}_{n+3}) \mp (\mathcal{FH}_m + i\mathcal{FH}_{m+1} + j\mathcal{FH}_{m+2} + k\mathcal{FH}_{m+3}) \\ &= (\mathcal{FH}_n \mp \mathcal{FH}_m) + i(\mathcal{FH}_{n+1} \mp \mathcal{FH}_{m+1}) + j(\mathcal{FH}_{n+2} \mp \mathcal{FH}_{m+2}) + k(\mathcal{FH}_{n+3} \mp \mathcal{FH}_{m+3}), \\ \mathcal{FHQ}_n \mp \mathcal{FHQ}_m &= (\mathcal{FQ}_n + \mathbf{i}\mathcal{FQ}_{n+1} + \varepsilon\mathcal{FQ}_{n+2} + \mathbf{h}\mathcal{FQ}_{n+3}) \mp (\mathcal{FQ}_m + \mathbf{i}\mathcal{FQ}_{m+1} + \varepsilon\mathcal{FQ}_{m+2} + \mathbf{h}\mathcal{FQ}_{m+3}) \\ &= (\mathcal{FQ}_n \mp \mathcal{FQ}_m) + \mathbf{i}(\mathcal{FQ}_{n+1} \mp \mathcal{FQ}_{m+1}) + \varepsilon(\mathcal{FQ}_{n+2} \mp \mathcal{FQ}_{m+2}) + \mathbf{h}(\mathcal{FQ}_{n+3} \mp \mathcal{FQ}_{m+3}). \end{aligned}$$

Definition 3.6. Multiplication of the Fibonacci hybrid quaternions is defined in terms of Fibonacci hybrid numbers $(\mathcal{FH}_n, \mathcal{FH}_m)$ as follows:

$$\begin{aligned}\mathcal{FHQ}_n\mathcal{FHQ}_m &= (\mathcal{FH}_n + i\mathcal{FH}_{n+1} + j\mathcal{FH}_{n+2} + k\mathcal{FH}_{n+3})(\mathcal{FH}_m + i\mathcal{FH}_{m+1} + j\mathcal{FH}_{m+2} + k\mathcal{FH}_{m+3})\\ &= (\mathcal{FH}_n\mathcal{FH}_m - \mathcal{FH}_{n+1}\mathcal{FH}_{m+1} - \mathcal{FH}_{n+2}\mathcal{FH}_{m+2} - \mathcal{FH}_{n+3}\mathcal{FH}_{m+3})\\ &\quad + i(\mathcal{FH}_n\mathcal{FH}_{m+1} + \mathcal{FH}_{n+1}\mathcal{FH}_m + \mathcal{FH}_{n+2}\mathcal{FH}_{m+3} - \mathcal{FH}_{n+3}\mathcal{FH}_{m+2})\\ &\quad + j(\mathcal{FH}_n\mathcal{FH}_{m+2} - \mathcal{FH}_{n+1}\mathcal{FH}_{m+3} + \mathcal{FH}_{n+2}\mathcal{FH}_m + \mathcal{FH}_{n+3}\mathcal{FH}_{m+1})\\ &\quad + k(\mathcal{FH}_n\mathcal{FH}_{m+3} + \mathcal{FH}_{n+1}\mathcal{FH}_{m+2} - \mathcal{FH}_{n+2}\mathcal{FH}_{m+1} + \mathcal{FH}_{n+3}\mathcal{FH}_m)\end{aligned}$$

or in terms of Fibonacci quaternions $(\mathcal{FQ}_n, \mathcal{FQ}_m)$ it can be defined as follows:

$$\begin{aligned}\mathcal{FHQ}_n\mathcal{FHQ}_m &= (\mathcal{FQ}_n + \mathbf{i}\mathcal{FQ}_{n+1} + \boldsymbol{\varepsilon}\mathcal{FQ}_{n+2} + \mathbf{h}\mathcal{FQ}_{n+3})(\mathcal{FQ}_m + \mathbf{i}\mathcal{FQ}_{m+1} + \boldsymbol{\varepsilon}\mathcal{FQ}_{m+2} + \mathbf{h}\mathcal{FQ}_{m+3})\\ &= (\mathcal{FQ}_n\mathcal{FQ}_m - \mathcal{FQ}_{n+1}\mathcal{FQ}_{m+1} + \mathcal{FQ}_{n+3}\mathcal{FQ}_{m+3} + \mathcal{FQ}_{n+1}\mathcal{FQ}_{m+2} + \mathcal{FQ}_{n+2}\mathcal{FQ}_{m+1})\\ &\quad + \mathbf{i}(\mathcal{FQ}_n\mathcal{FQ}_{m+1} + \mathcal{FQ}_{n+1}\mathcal{FQ}_m + \mathcal{FQ}_{n+1}\mathcal{FQ}_{m+3} - \mathcal{FQ}_{n+3}\mathcal{FQ}_{m+1})\\ &\quad + \boldsymbol{\varepsilon}(\mathcal{FQ}_n\mathcal{FQ}_{m+2} + \mathcal{FQ}_{n+1}\mathcal{FQ}_{m+3} + \mathcal{FQ}_{n+2}\mathcal{FQ}_m - \mathcal{FQ}_{n+2}\mathcal{FQ}_{m+3}\\ &\qquad - \mathcal{FQ}_{n+3}\mathcal{FQ}_{m+1} + \mathcal{FQ}_{n+3}\mathcal{FQ}_{m+2})\\ &\quad + \mathbf{h}(\mathcal{FQ}_n\mathcal{FQ}_{m+3} - \mathcal{FQ}_{n+1}\mathcal{FQ}_{m+2} + \mathcal{FQ}_{n+2}\mathcal{FQ}_{m+1} + \mathcal{FQ}_{n+3}\mathcal{FQ}_m).\end{aligned}$$

The scalar and vector parts of $\mathcal{FHQ}_n$ which is the nth term of the Fibonacci hybrid quaternion sequence $(\mathcal{FHQ}_n)$ are denoted by

$$S_{\mathcal{FHQ}_n} = \mathcal{FH}_n \quad \text{and} \quad V_{\mathcal{FHQ}_n} = i\mathcal{FH}_{n+1} + j\mathcal{FH}_{n+2} + k\mathcal{FH}_{n+3}$$

So, any Fibonacci hybrid quaternion $\mathcal{FHQ}_n$ can be written as $\mathcal{FHQ}_n = S_{\mathcal{FHQ}_n} + V_{\mathcal{FHQ}_n}$. Now we can redefine addition and subtraction as

$$\begin{aligned}\mathcal{FHQ}_n \mp \mathcal{FHQ}_m &= (S_{\mathcal{FHQ}_n} + V_{\mathcal{FHQ}_n}) \mp (S_{\mathcal{FHQ}_m} + V_{\mathcal{FHQ}_m})\\ &= (S_{\mathcal{FHQ}_n} \mp S_{\mathcal{FHQ}_m}) + (V_{\mathcal{FHQ}_n} \mp V_{\mathcal{FHQ}_m}),\end{aligned}$$

and multiplication

$$\begin{aligned}\mathcal{FHQ}_n\mathcal{FHQ}_n &= (S_{\mathcal{FHQ}_n} + V_{\mathcal{FHQ}_n})(S_{\mathcal{FHQ}_n} + V_{\mathcal{FHQ}_n})\\ &= S_{\mathcal{FHQ}_n}S_{\mathcal{FHQ}_n} - \langle V_{\mathcal{FHQ}_n}, V_{\mathcal{FHQ}_n}\rangle\\ &\quad + S_{\mathcal{FHQ}_n}V_{\mathcal{FHQ}_n} + S_{\mathcal{FHQ}_n}V_{\mathcal{FHQ}_n} + V_{\mathcal{FHQ}_n} \times V_{\mathcal{FHQ}_n}.\end{aligned}$$

Definition 3.7. The conjugate of Fibonacci hybrid quaternion can be define three different types for $\mathcal{FHQ}_n = \mathcal{FQ}_n + \mathbf{i}\mathcal{FQ}_{n+1} + \boldsymbol{\varepsilon}\mathcal{FQ}_{n+2} + \mathbf{h}\mathcal{FQ}_{n+3}$

i) Quaternion conjugate, $\overline{\mathcal{FHQ}_n}$: $\overline{\mathcal{FHQ}_n} = \overline{\mathcal{FQ}}_n + \mathbf{i}\overline{\mathcal{FQ}}_{n+1} + \boldsymbol{\varepsilon}\overline{\mathcal{FQ}}_{n+2} + \mathbf{h}\overline{\mathcal{FQ}}_{n+3}$,

ii) Hybrid conjugate, $(\mathcal{FHQ}_n)^C$: $(\mathcal{FHQ}_n)^C = \mathcal{FQ}_n - \mathbf{i}\mathcal{FQ}_{n+1} - \boldsymbol{\varepsilon}\mathcal{FQ}_{n+2} - \mathbf{h}\mathcal{FQ}_{n+3}$,

iii) Total conjugate, $(\mathcal{FHQ}_n)^\dagger$: $(\mathcal{FHQ}_n)^\dagger = \overline{(\mathcal{FHQ}_n)^C} = \overline{\mathcal{FQ}}_n - \mathbf{i}\overline{\mathcal{FQ}}_{n+1} - \boldsymbol{\varepsilon}\overline{\mathcal{FQ}}_{n+2} - \mathbf{h}\overline{\mathcal{FQ}}_{n+3}$.

Theorem 3.8. Let $\mathcal{FHQ}_n$ be nth term of the Fibonacci sequence. Then, for $n \geq 1$ we can give the following relations:

i) $\mathcal{FHQ}_n + \mathcal{FHQ}_{n+1} = \mathcal{FHQ}_{n+2}$,

ii) $\mathcal{FHQ}_n - i\mathcal{FHQ}_{n+1} - j\mathcal{FHQ}_{n+2} - k\mathcal{FHQ}_{n+3} = \mathcal{FH}_n + \mathcal{FH}_{n+2} + \mathcal{FH}_{n+4} + \mathcal{FH}_{n+6}$
$= \mathcal{LH}_{n+1} + \mathcal{LH}_{n+5},$

iii) $\mathcal{FHQ}_n - \mathbf{i}\mathcal{FHQ}_{n+1} - \varepsilon\mathcal{FHQ}_{n+2} - \mathbf{h}\mathcal{FHQ}_{n+3} = \mathcal{FQ}_n - \mathcal{FQ}_{n+2} - 2\mathcal{FQ}_{n+3} + \mathcal{FQ}_{n+6}.$

Proof. Let $\mathcal{FH}_n$, $\mathcal{LH}_n$ and $\mathcal{FQ}_n$ be nth Fibonacci hybrid number, nth Lucas hybrid number and nth Fibonacci quaternion, respectively.
i) We can show this equality in two ways; first one is using Fibonacci hybrid numbers:

$$\begin{aligned}\mathcal{FHQ}_n + \mathcal{FHQ}_{n+1} &= (\mathcal{FH}_n + i\mathcal{FH}_{n+1} + j\mathcal{FH}_{n+2} + k\mathcal{FH}_{n+3})\\ &\quad + (\mathcal{FH}_{n+1} + i\mathcal{FH}_{n+2} + j\mathcal{FH}_{n+3} + k\mathcal{FH}_{n+4})\\ &= (\mathcal{FH}_n + \mathcal{FH}_{n+1}) + i(\mathcal{FH}_{n+1} + \mathcal{FH}_{n+2})\\ &\quad + j(\mathcal{FH}_{n+2} + \mathcal{FH}_{n+3}) + k(\mathcal{FH}_{n+3} + \mathcal{FH}_{n+4})\\ &= \mathcal{FH}_{n+2} + i\mathcal{FH}_{n+3} + j\mathcal{FH}_{n+4} + k\mathcal{FH}_{n+5} = \mathcal{FHQ}_{n+2}.\end{aligned}$$

The second one is using Fibonacci quaternions:

$$\begin{aligned}\mathcal{FHQ}_n + \mathcal{FHQ}_{n+1} &= (\mathcal{FQ}_n + \mathbf{i}\mathcal{FQ}_{n+1} + \varepsilon\mathcal{FQ}_{n+2} + \mathbf{h}\mathcal{FQ}_{n+3})\\ &\quad + (\mathcal{FQ}_{n+1} + \mathbf{i}\mathcal{FQ}_{n+2} + \varepsilon\mathcal{FQ}_{n+3} + \mathbf{h}\mathcal{FQ}_{n+4})\\ &= (\mathcal{FQ}_n + \mathcal{FQ}_{n+1}) + \mathbf{i}(\mathcal{FQ}_{n+1} + \mathcal{FQ}_{n+2}\\ &\quad + \varepsilon(\mathcal{FQ}_{n+2} + \mathcal{FQ}_{n+3}) + \mathbf{h}(\mathcal{FQ}_{n+3} + \mathcal{FQ}_{n+4})\\ &= \mathcal{FQ}_{n+2} + \mathbf{i}\mathcal{FQ}_{n+3} + \varepsilon\mathcal{FQ}_{n+4} + \mathbf{h}\mathcal{FQ}_{n+5} = \mathcal{FHQ}_{n+2}.\end{aligned}$$

ii) If we use $\mathcal{FH}_{n-1} + \mathcal{FH}_{n+1} = \mathcal{LH}_n$ [11], we have

$$\begin{aligned}\mathcal{FHQ}_n - i\mathcal{FHQ}_{n+1} - j\mathcal{FHQ}_{n+2} - k\mathcal{FHQ}_{n+3} &= (\mathcal{FH}_n + i\mathcal{FH}_{n+1} + j\mathcal{FH}_{n+2} + k\mathcal{FH}_{n+3})\\ &\quad - i(\mathcal{FH}_{n+1} + i\mathcal{FH}_{n+2} + j\mathcal{FH}_{n+3} + k\mathcal{FH}_{n+4})\\ &\quad - j(\mathcal{FH}_{n+2} + i\mathcal{FH}_{n+3} + j\mathcal{FH}_{n+4} + k\mathcal{FH}_{n+5})\\ &\quad - k(\mathcal{FH}_{n+3} + i\mathcal{FH}_{n+4} + j\mathcal{FH}_{n+5} + k\mathcal{FH}_{n+6})\\ &= \mathcal{FH}_n + \mathcal{FH}_{n+2} + \mathcal{FH}_{n+4} + \mathcal{FH}_{n+6}\\ &= \mathcal{LH}_{n+1} + \mathcal{LH}_{n+5}.\end{aligned}$$

iii)
$$\begin{aligned}&\mathcal{FHQ}_n - \mathbf{i}\mathcal{FHQ}_{n+1} - \varepsilon\mathcal{FHQ}_{n+2} - h\mathcal{FHQ}_{n+3}\\ &\quad= (\mathcal{FHQ}_n + \mathbf{i}\mathcal{FHQ}_{n+1} + \varepsilon\mathcal{FHQ}_{n+2} + \mathbf{h}\mathcal{FHQ}_{n+3})\\ &\qquad -\mathbf{i}(\mathcal{FHQ}_{n+1} + \mathbf{i}\mathcal{FHQ}_{n+2} + \varepsilon\mathcal{FHQ}_{n+3} + \mathbf{h}\mathcal{FHQ}_{n+4})\\ &\qquad -\varepsilon(\mathcal{FHQ}_{n+2} + \mathbf{i}\mathcal{FHQ}_{n+3} + \varepsilon\mathcal{FHQ}_{n+4} + \mathbf{h}\mathcal{FHQ}_{n+5})\\ &\qquad -\mathbf{h}(\mathcal{FHQ}_{n+3} + \mathbf{i}\mathcal{FHQ}_{n+4} + \varepsilon\mathcal{FHQ}_{n+5} + \mathbf{h}\mathcal{FHQ}_{n+6})\\ &\quad= \mathcal{FHQ}_n - \mathcal{FHQ}_{n+2} - 2\,\mathcal{FHQ}_{n+3} + \mathcal{FHQ}_{n+6}.\end{aligned}$$

Q.E.D.

Theorem 3.9. Let $\mathcal{FHQ}_n$ and $\mathcal{LHQ}_n$ be *nth* Fibonacci hybrid quaternion and the Lucas hybrid quaternion sequences respectively. The following relations are satisfied:

i) $\mathcal{FHQ}_{n-1}+\mathcal{FHQ}_{n+1}=\mathcal{LHQ}_n$

ii) $\mathcal{FHQ}_{n+2}-\mathcal{FHQ}_{n-2}=\mathcal{LHQ}_n$

Proof.

i) We know that $\mathcal{LH}_n=\mathcal{FH}_{n-1}+\mathcal{FH}_{n+1}$. Then we have

$$\begin{aligned}\mathcal{FHQ}_{n-1}+\mathcal{FHQ}_{n+1}&=(\mathcal{FH}_{n-1}+i\mathcal{FH}_n+j\mathcal{FH}_{n+1}+k\mathcal{FH}_{n+2})\\&\quad+(\mathcal{FH}_{n+1}+i\mathcal{FH}_{n+2}+j\mathcal{FH}_{n+3}+k\mathcal{FH}_{n+4})\\&=(\mathcal{FH}_{n-1}+\mathcal{FH}_{n+1})+i(\mathcal{FH}_n+F_{n+2})\\&\quad+j(\mathcal{FH}_{n+1}+\mathcal{FH}_{n+3})+k(\mathcal{FH}_{n+2}+\mathcal{FH}_{n+4})\\&=\mathcal{LH}_n+i\mathcal{LH}_{n+1}+j\mathcal{LH}_{n+2}+k\mathcal{LH}_{n+3}\\&=\mathcal{LHQ}_n\end{aligned}$$

ii) We know that $\mathcal{FH}_n=\mathcal{FH}_{n+2}-\mathcal{FH}_{n-2}$.

$$\begin{aligned}\mathcal{FHQ}_{n+2}-\mathcal{FHQ}_{n-2}&=(\mathcal{FH}_{n+2}+i\mathcal{FH}_{n+3}+j\mathcal{FH}_{n+4}+k\mathcal{FH}_{n+5})\\&\quad-(\mathcal{FH}_{n-2}+i\mathcal{FH}_{n-1}+j\mathcal{FH}_n+k\mathcal{FH}_{n+1})\\&=(\mathcal{FH}_{n+2}-\mathcal{FH}_{n-2})+i(\mathcal{FH}_{n+3}-\mathcal{FH}_{n-1})\\&\quad+j(\mathcal{FH}_{n+4}-\mathcal{FH}_n)+k(\mathcal{FH}_{n+5}-\mathcal{FH}_{n+1})\\&=\mathcal{LH}_n+i\mathcal{LH}_{n+1}+j\mathcal{LH}_{n+2}+k\mathcal{LH}_{n+3}\\&=\mathcal{LHQ}_n\end{aligned}$$

Q.E.D.

Theorem 3.10. Let $\mathcal{FHQ}_n$ be *nth* Fibonacci hybrid quaternion sequence $(\mathcal{FHQ}_n)$ and $\overline{\mathcal{FHQ}_n}$, $\mathcal{FHQ}_n^C$, $\mathcal{FHQ}_n^\dagger$ be the quaternion conjugate, hybrid conjugate and total conjugate of *nth* Fibonacci hybrid quaternion respectively. The following relations are satisfies:

i) $\mathcal{FHQ}_n+\overline{\mathcal{FHQ}_n}=2\mathcal{FH}_n$

ii) $\mathcal{FHQ}_n+\mathcal{FHQ}_n^C=2\mathcal{FQ}_n$

iii) $\mathcal{FHQ}_n+\mathcal{FHQ}_n^\dagger=-2\mathcal{F}_n-8\mathcal{F}_{n+1}+2(\mathcal{FHQ}_{n+1}+\mathcal{FHQ}_{n+2}+\mathcal{FHQ}_{n+3})$

Proof.

i)
$$\begin{aligned}\mathcal{FHQ}_n+\overline{\mathcal{FHQ}_n}&=(\mathcal{FH}_n+i\mathcal{FH}_{n+1}+j\mathcal{FH}_{n+2}+k\mathcal{FH}_{n+3})\\&\quad+(\mathcal{FH}_n-i\mathcal{FH}_{n+1}-j\mathcal{FH}_{n+2}-k\mathcal{FH}_{n+3})\\&=2\mathcal{FH}_n\end{aligned}$$

ii)
$$\begin{aligned}\mathcal{FHQ}_n+\mathcal{FHQ}_n^C&=(\mathcal{FH}_n+i\mathcal{FH}_{n+1}+j\mathcal{FH}_{n+2}+k\mathcal{FH}_{n+3})\\&\quad+(\mathcal{FH}_n^C+i\mathcal{FH}_{n+1}^C+j\mathcal{FH}_{n+2}^C+k\mathcal{FH}_{n+3}^C)\\&=(\mathcal{FH}_n+\mathcal{FH}_n^C)+i(\mathcal{FH}_{n+1}+\mathcal{FH}_{n+1}^C)\\&\quad+j(\mathcal{FH}_{n+2}+\mathcal{FH}_{n+2}^C)+k(\mathcal{FH}_{n+3}+\mathcal{FH}_{n+3}^C)\\&=2(\mathcal{F}_n+i\mathcal{F}_{n+1}+j\mathcal{F}_{n+2}k\mathcal{F}_{n+3})=2\mathcal{FQ}_n\end{aligned}$$

iii) $$\begin{aligned}\mathcal{FHQ}_n + \mathcal{FHQ}_n^{\dagger} &= (\mathcal{FH}_n + i\mathcal{FH}_{n+1} + j\mathcal{FH}_{n+2} + k\mathcal{FH}_{n+3}) \\ &\quad + (\mathcal{FH}_n^C - i\mathcal{FH}_{n+1}^C - j\mathcal{FH}_{n+2}^C - k\mathcal{FH}_{n+3}^C) \\ &= (\mathcal{FH}_n + \mathcal{FH}_n^C) + i(\mathcal{FH}_{n+1} - \mathcal{FH}_{n+1}^C) \\ &\quad + j(\mathcal{FH}_{n+2} - \mathcal{FH}_{n+2}^C) + k(\mathcal{FH}_{n+3} - \mathcal{FH}_{n+3}^C) \\ &= -2\mathcal{F}_n - 8\mathcal{F}_{n+1} + 2(\mathcal{FH}_{n+1} + \mathcal{FH}_{n+2} + \mathcal{FH}_{n+3})\end{aligned}$$

Q.E.D.

Theorem 3.11 (Binet's Formulas)**.** Let $\mathcal{FHQ}_n$ and $\mathcal{LHQ}_n$ be Fibonacci hybrid quaternion and Lucas hybrid quaternion respectively. The Binet formulas for these hybrid quaternions are given as follows:

i) $\mathcal{FHQ}_n = \dfrac{\alpha^* \underline{\alpha}\alpha^n - \beta^* \underline{\beta}\beta^n}{\alpha - \beta}$

ii) $\mathcal{LHQ}_n = \alpha^* \underline{\alpha}\alpha^n + \beta^* \underline{\beta}\beta^n$

where $\alpha^* = 1 + \mathbf{i}\alpha + \boldsymbol{\varepsilon}\alpha^2 + \mathbf{h}\alpha^3$, $\beta^* = 1 + \mathbf{i}\beta + \boldsymbol{\varepsilon}\beta^2 + \mathbf{h}\beta^3$, $\underline{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3$ and $\underline{\beta} = 1 + i\beta + j\beta^2 + k\beta^3$.

Proof. In [14], Halici gave the Binet's formula for Fibonacci and Lucas quaternions by

$$\mathcal{FQ}_n = \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{\alpha - \beta} \tag{3.8}$$

and

$$\mathcal{LQ}_n = \underline{\alpha}\alpha^n + \underline{\beta}\beta^n. \tag{3.9}$$

i) By using (3.8), we have

$$\begin{aligned}\mathcal{FHQ}_n &= \mathcal{FQ}_n + \mathbf{i}\mathcal{FQ}_{n+1} + \boldsymbol{\varepsilon}\mathcal{FQ}_{n+2} + \mathbf{h}\mathcal{FQ}_{n+3} \\ &= \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{\alpha - \beta} + \mathbf{i}\frac{\underline{\alpha}\alpha^{n+1} - \underline{\beta}\beta^{n+1}}{\alpha - \beta} + \boldsymbol{\varepsilon}\frac{\underline{\alpha}\alpha^{n+2} - \underline{\beta}\beta^{n+2}}{\alpha - \beta} + \mathbf{h}\frac{\underline{\alpha}\alpha^{n+3} - \underline{\beta}\beta^{n+3}}{\alpha - \beta} \\ &= \frac{(\underline{\alpha}\alpha^n - \underline{\beta}\beta^n) + \mathbf{i}(\underline{\alpha}\alpha^{n+1} - \underline{\beta}\beta^{n+1}) + \boldsymbol{\varepsilon}(\underline{\alpha}\alpha^{n+2} - \underline{\beta}\beta^{n+2}) + \mathbf{h}(\underline{\alpha}\alpha^{n+3} - \underline{\beta}\beta^{n+3})}{\alpha - \beta} \\ &= \frac{(\underline{\alpha}\alpha^n + \mathbf{i}\underline{\alpha}\alpha^{n+1} + \boldsymbol{\varepsilon}\underline{\alpha}\alpha^{n+2} + \mathbf{h}\underline{\alpha}\alpha^{n+3}) - (\underline{\beta}\beta^n + \mathbf{i}\underline{\beta}\beta^{n+1} + \boldsymbol{\varepsilon}\underline{\beta}\beta^{n+2} + \mathbf{h}\underline{\beta}\beta^{n+3})}{\alpha - \beta} \\ &= \frac{\underline{\alpha}\alpha^n(1 + \mathbf{i}\alpha + \boldsymbol{\varepsilon}\alpha^2 + \mathbf{h}\alpha^3) - \underline{\beta}\beta^n(1 + \mathbf{i}\beta + \boldsymbol{\varepsilon}\beta^2 + \mathbf{h}\beta^3)}{\alpha - \beta} \\ &= \frac{\alpha^* \underline{\alpha}\alpha^n - \beta^* \underline{\alpha}\beta^n}{\alpha - \beta}.\end{aligned}$$

ii) By using (3.9), we have

$$\begin{aligned}
\mathcal{LHQ}_n &= \mathcal{LQ}_n + \mathbf{i}\mathcal{LQ}_{n+1} + \boldsymbol{\varepsilon}\mathcal{LQ}_{n+2} + h\mathcal{LQ}_{n+3} \\
&= \underline{\alpha}\alpha^n + \underline{\beta}\beta^n + \mathbf{i}\underline{\alpha}\alpha^{n+1} + \underline{\beta}\beta^{n+1} + \boldsymbol{\varepsilon}\underline{\alpha}\alpha^{n+2} + \underline{\beta}\beta^{n+2} + \mathbf{h}\underline{\alpha}\alpha^{n+3} + \underline{\beta}\beta^{n+3} \\
&= (\underline{\alpha}\alpha^n + \underline{\beta}\beta^n) + \mathbf{i}(\underline{\alpha}\alpha^{n+1} + \underline{\beta}\beta^{n+1}) + \boldsymbol{\varepsilon}(\underline{\alpha}\alpha^{n+2} + \underline{\beta}\beta^{n+2}) + \mathbf{h}(\underline{\alpha}\alpha^{n+3} + \underline{\beta}\beta^{n+3}) \\
&= (\underline{\alpha}\alpha^n + \mathbf{i}\underline{\alpha}\alpha^{n+1} + \boldsymbol{\varepsilon}\underline{\alpha}\alpha^{n+2} + \mathbf{h}\underline{\alpha}\alpha^{n+3}) + (\underline{\beta}\beta^n + \mathbf{i}\underline{\beta}\beta^{n+1} + \boldsymbol{\varepsilon}\underline{\beta}\beta^{n+2} + \mathbf{h}\underline{\beta}\beta^{n+3}) \\
&= \underline{\alpha}\alpha^n(1 + \mathbf{i}\alpha + \boldsymbol{\varepsilon}\alpha^2 + \mathbf{h}\alpha^3) + \underline{\beta}\beta^n(1 + \mathbf{i}\beta + \boldsymbol{\varepsilon}\beta^2 + \mathbf{h}\beta^3) \\
&= \alpha^*\underline{\alpha}\alpha^n + \beta^*\underline{\beta}\beta^n.
\end{aligned}$$

Q.E.D.

Theorem 3.12 (Cassini's Identities)**.** The following equations are hold:

$$C_1 = \mathcal{FHQ}_{n+1}\mathcal{FHQ}_{n-1} - \mathcal{FHQ}_n^2 = (-1)^n \frac{\alpha\alpha^*\beta^*\underline{\alpha\beta} - \beta\beta^*\alpha^*\underline{\beta\alpha}}{\alpha - \beta},$$

$$C_2 = \mathcal{LHQ}_{n+1}\mathcal{LHQ}_{n-1} - \mathcal{LHQ}_n^2 = (-1)^n\sqrt{5}(\alpha\alpha^*\beta^*\underline{\alpha\beta} - \beta\beta^*\alpha^*\underline{\beta\alpha})$$

where α and β be the roots of equation $x^2-2x-1=0$, $\underline{\alpha} = 1+\alpha i+\alpha^2 j+\alpha^3 L$, $\underline{\beta} = 1+\beta i+\beta^2 j+\beta^3 k$, $\alpha^* = 1 + \alpha\mathbf{i} + \alpha^2\boldsymbol{\varepsilon} + \alpha^3\mathbf{h}$ and $\beta^* = 1 + \beta\mathbf{i} + \beta^2\boldsymbol{\varepsilon} + \beta^3\mathbf{h}$.

Proof. For the first Cassini identity C_1, we get

$$\begin{aligned}
C_1 &= \left(\frac{\alpha^{n+1}\alpha^*\underline{\alpha} - \beta^{n+1}\beta^*\underline{\beta}}{\alpha - \beta}\right)\left(\frac{\alpha^{n-1}\alpha^*\underline{\alpha} - \beta^{n-1}\beta^*\underline{\beta}}{\alpha - \beta}\right) - \left(\frac{\alpha^n\alpha^*\underline{\alpha} - \beta^n\beta^*\underline{\beta}}{\alpha - \beta}\right)^2 \\
&= \frac{\alpha^{2n}(\alpha^*)^2\underline{\alpha}^2 - \alpha^{n+1}\beta^{n-1}\alpha^*\beta^*\underline{\alpha\beta} - \beta^{n+1}\alpha^{n-1}\beta^*\alpha^*\underline{\beta\alpha} + \beta^{2n}(\beta^*)^2\beta^2}{(\alpha-\beta)^2} \\
&\quad - \frac{\alpha^{2n}(\alpha^*)^2\underline{\alpha}^2 - \alpha^n\beta^n\alpha^*\beta^*\underline{\alpha\beta} - \beta^n\alpha^n\beta^*\alpha^*\underline{\beta\alpha} + \beta^{2n}(\beta^*)^2\beta^2}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{n-1}\beta^n\beta^*\alpha^*(\alpha-\beta)\underline{\beta\alpha} - \alpha^n\beta^{n-1}\alpha^*\beta^*(\alpha-\beta)\underline{\alpha\beta}}{(\alpha-\beta)^2} \\
&= (-1)^n\frac{(\alpha\alpha^*\beta^*\underline{\alpha\beta} - \beta\beta^*\alpha^*\underline{\beta\alpha})}{\alpha-\beta}
\end{aligned}$$

For the second Cassini identity C_2, we get

$$\begin{aligned}
C_2 &= (\alpha^{n+1}\alpha^*\underline{\alpha} + \beta^{n+1}\beta^*\underline{\beta})(\alpha^{n-1}\alpha^*\underline{\alpha} + \beta^{n-1}\beta^*\underline{\beta}) - (\alpha^n\alpha^*\underline{\alpha} + \beta^n\beta^*\underline{\beta})^2 \\
&= \alpha^{2n}(\alpha^*)^2\underline{\alpha}^2 + \alpha^{n+1}\beta^{n-1}\alpha^*\beta^*\underline{\alpha\beta} + \beta^{n+1}\alpha^{n-1}\beta^*\alpha^*\underline{\beta\alpha} + \beta^{2n}(\beta^*)^2\beta^2 \\
&\quad - \alpha^{2n}(\alpha^*)^2\underline{\alpha}^2 + \alpha^n\beta^n\alpha^*\beta^*\underline{\alpha\beta} + \beta^n\alpha^n\beta^*\alpha^*\underline{\beta\alpha} + \beta^{2n}(\beta^*)^2\beta^2 \\
&= \alpha^{n-1}\beta^n\beta^*\alpha^*(\beta-\alpha)\underline{\beta\alpha} + \alpha^n\beta^{n-1}\alpha^*\beta^*(\alpha-\beta)\underline{\alpha\beta} \\
&= \alpha^{n-1}\beta^{n-1}(\alpha-\beta)(\alpha\alpha^*\beta^*\underline{\alpha\beta} - \beta\beta^*\alpha^*\underline{\beta\alpha}) \\
&= (-1)^n\sqrt{5}(\alpha\alpha^*\beta^*\underline{\alpha\beta} - \beta\beta^*\alpha^*\underline{\beta\alpha}).
\end{aligned}$$

Q.E.D.

4 Conclusion

In this paper, we have introduced the Horadam hybrid quaternions and some special classes of number sequences such as Fibonacci, Lucas, Pell and Jacobsthal hybrid quaternions. Especially, we have examined the Fibonacci and Lucas hybrid quaternions comprehensively. Moreover, some identities like Binet's formula and Cassini's identity for Fibonacci and Lucas hybrid quaternions have been given. Other sequences on Hybrid quaternions can be studied as future works.

References

[1] W. R. Hamilton, *Elements of Quaternions*, Longmans , Green and Co., London (1866)

[2] E.Pervin and J. A. Webb, *Quaternions in computer vision and robotics*, (No. CMU-CS-82-150) Carnegie-Mellon Univ. Pittsburgh pa Dept. Of Computer Science. (1982)

[3] S. Qiao, Q. Liao, S. Wei and H. J. Su, *Inverse kinematic analysis of the general 6R serial manipulators based on double quaternions*, Mechanism and Machine Theory, **45(2)** (2010) 193–199.

[4] J. Vince, *Quaternions for computer graphics*, Springer Science and Business Media (2011)

[5] F. Zhang, *Quaternions and matrices of quaternions*, Linear algebra and its applications **251** (1997) 21–57.

[6] M. Özdemir, *Introduction to hybrid numbers*, Adv. Appl. Clifford Algebras **28(1)** (2018) 1–32.

[7] A. Dağdeviren, *Hybrid Quaternions* (2021) Submitted.

[8] A. F. Horadam, *A generalized Fibonacci sequence*, The American Mathematical Monthly **68(5)** (1961) 455–459.

[9] A. F. Horadam, *Basic properties of a certain generalized sequence of numbers*, The Fibonacci Quarterly **3(3)** (1965) 161–176.

[10] C. B. Çimen and A. İpek, *On pell quaternions and Pell-Lucas quaternions*, Adv. Appl. Clifford Algebras **26(1)** (2016) 39–51.

[11] A. Szynal-Liana, *The Horadam hybrid numbers*, Discussiones Mathematicae-General Algebra and Applications **38(1)** (2018) 91–98.

[12] F. Torunbalcı Aydın, *Generalized dual Pell quaternions*, Notes on Number Theory and Discrete Mathematics **23(4)** (2017) 66–84.

[13] S. Halıcı, *On complex Fibonacci quaternions*, Adv. Appl. Clifford Algebras **23(1)** (2013) 105–112.

[14] S. Halıcı, *On fibonacci quaternions*, Adv. Appl. Clifford Algebras **22(2)** (2012) 321–327.

[15] D. Tasci, *Padovan and Pell-Padovan Quaternions*, Journal of Science and Arts **18** (2018) 125–132.

[16] Y. Yazlik and N. Taskara, *A note on generalized k-Horadam sequence*, Computers and Mathematics with Applications **63(1)** (2012) 36–41

[17] O. D. Bagdasar and P. J. Larcombe, *On the number of complex Horadam sequences with a fixed period*, Fibonacci Quarterly **51(4)** (2013) 339–347.

[18] G. Cerda-Morales, *Investigation of generalized hybrid Fibonacci numbers and their properties*, arXiv preprint arXiv:1806.02231. (2018)

[19] M. Liana, A. Szynal-Liana and I. Wloch, *On Pell hybrinomials*, Miskolc Mathematical Notes, **20(2)** (2019) 1051–1062.

[20] A. Szynal-Liana and I. Wloch, *On Jacobsthal and Jacobsthal-Lucas Hybrid Numbers*, In Annales Mathematicae Silesianae **33(1)** (2019) 276–283.

[21] A. Szynal-Liana, I. Wloch, *The Fibonacci hybrid numbers*, Utilitas Math. **110** (2019) 3–10.

[22] E. Polatli, *Hybrid Numbers with Fibonacci and Lucas Hybrid Number Coefficients*, Preprints, 2020120349 (2020).

[23] M. Akyiğit, H. H. Koksal and M. Tosun, *Split fibonacci quaternions*, Adv. Appl. Clifford Algebras **23(3)** (2013) 535–545.